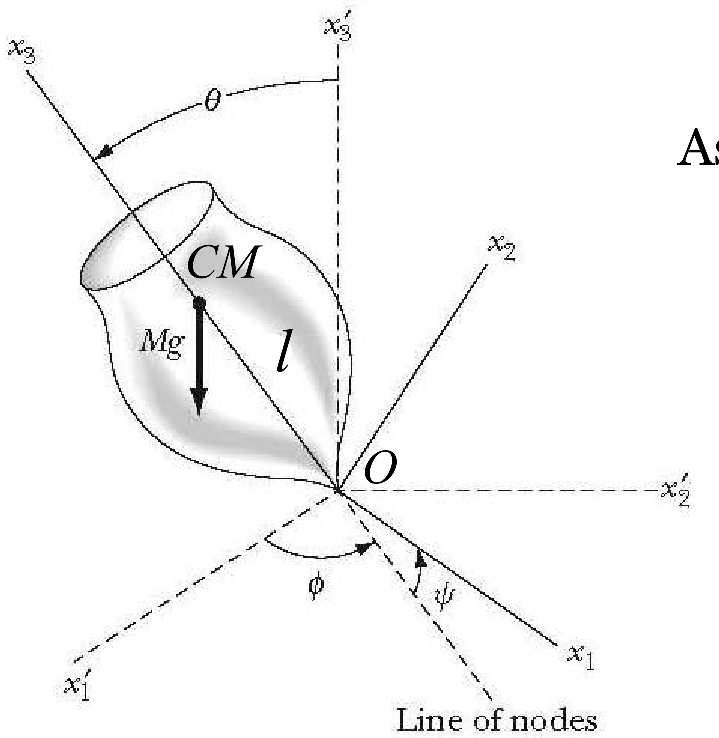
PHYS 705: Classical Mechanics

More Motion of Rigid Bodies

Symmetric Top in an Uniform Gravity Field

We have been looking at motion of torque-free rigid bodies.

Now, we consider a rigid body under the influence of gravity so that $U \neq 0$

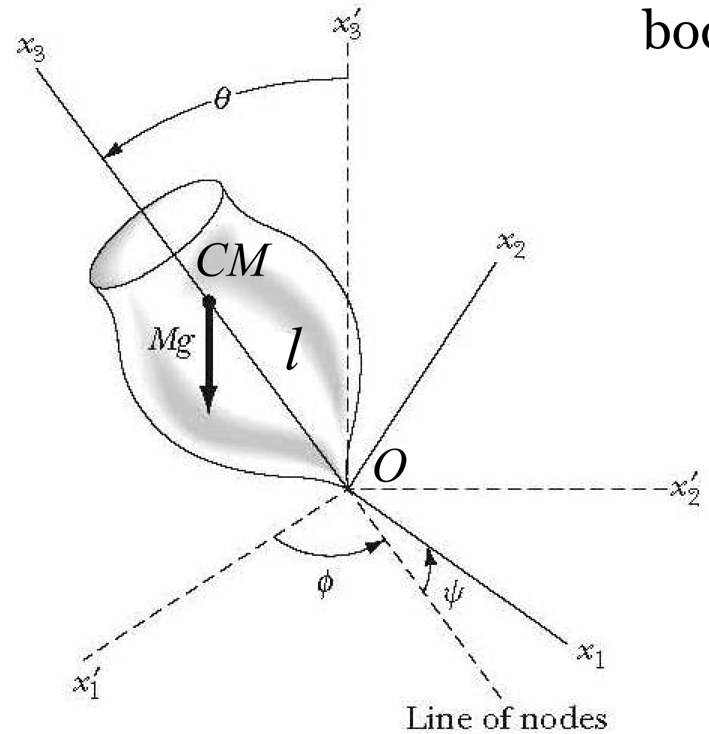


Assumptions:

- One point of the body remains fixed at the origin *O* but it not necessary coincides with the *CM*
- Again, we assume a symmetric top, i.e.,

$$I_1 = I_2 \neq I_3$$

Symmetric Top in an Uniform Gravity Field



U: We can treat the potential energy *U* as if the rigid body with all its mass *M* concentrated at the *CM*.

$$\Rightarrow \quad U = Mgl\cos\theta$$

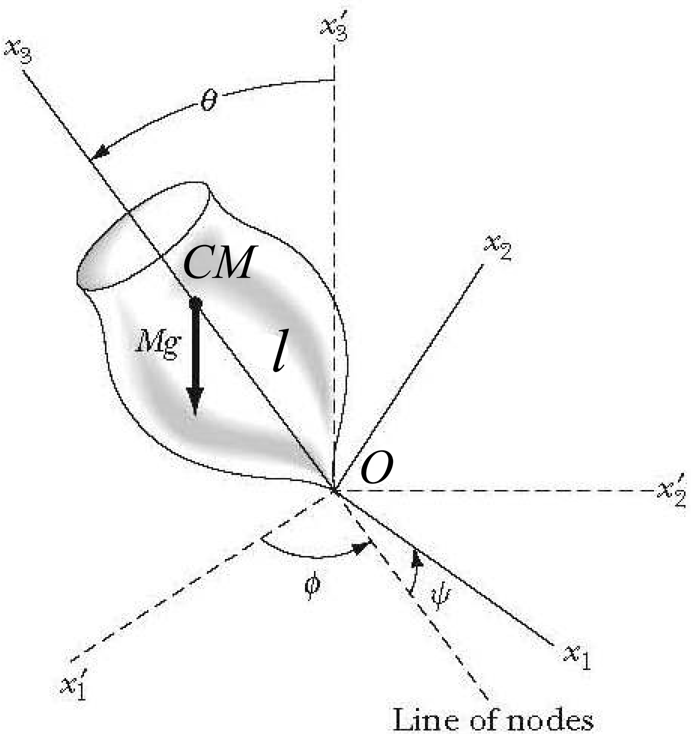
- $U = 0$ is taken to be at $x_3' = 0$ plane
- l is the dist from the fixed point (*O*) to *CM*

T: With *O* being fixed (not moving) in the fixed-frame and the body axes align with the Principal Axes,

$$T_{total} = T_{rot} = \frac{1}{2} I_i \omega_i^2$$

Symmetric Top in an Uniform Gravity Field

To analyze the motion in the body frame, we can use the Euler's eqs:



$$I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 = N_1$$
$$I_2\dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1 = N_2$$
$$I_3\dot{\omega}_3 = N_3 \longleftarrow \boxed{I_1 = I_2}$$

The Euler's equations provide a description for the time evolution of $(\omega_1, \omega_2, \omega_3)$ in the "body" axes but they are NOT necessary the most accessible variables such as the generalized coords: (ϕ, θ, ψ)

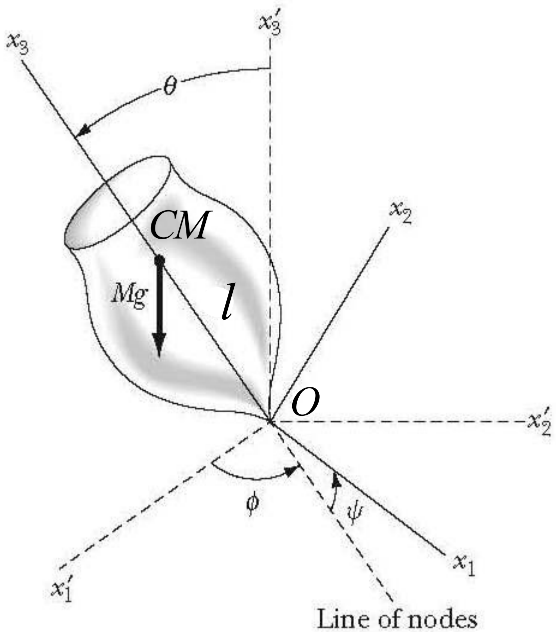
→ Alternatively, we can use the Lagrangian method to directly obtain EOM for $(\dot{\phi}, \dot{\theta}, \dot{\psi})$

Symmetric Top in an Uniform Gravity Field

We will write out the Lagrangian in terms of the generalized coordinates (the three Eulers' angles)

Again, we will align our body axes to coincide with the principal axes.

$$I_1 = I_2$$



$$T = \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right) = \frac{I_1}{2}\left(\omega_1^2 + \omega_2^2\right) + \frac{I_3}{2}\omega_3^2$$

Using $\boldsymbol{\omega} = \begin{pmatrix} \dot\phi \sin\psi \sin\theta + \dot\theta\cos\psi \\ \dot\phi\cos\psi\sin\theta - \dot\theta\sin\psi \\ \dot\phi\cos\theta + \dot\psi \end{pmatrix}$

$$\omega_1^2 = \dot\phi^2 \sin^2\psi \sin^2\theta + \dot\theta^2\cos^2\psi + \cancel{2\dot\phi\dot\theta\sin\psi\sin\theta\cos\psi}$$

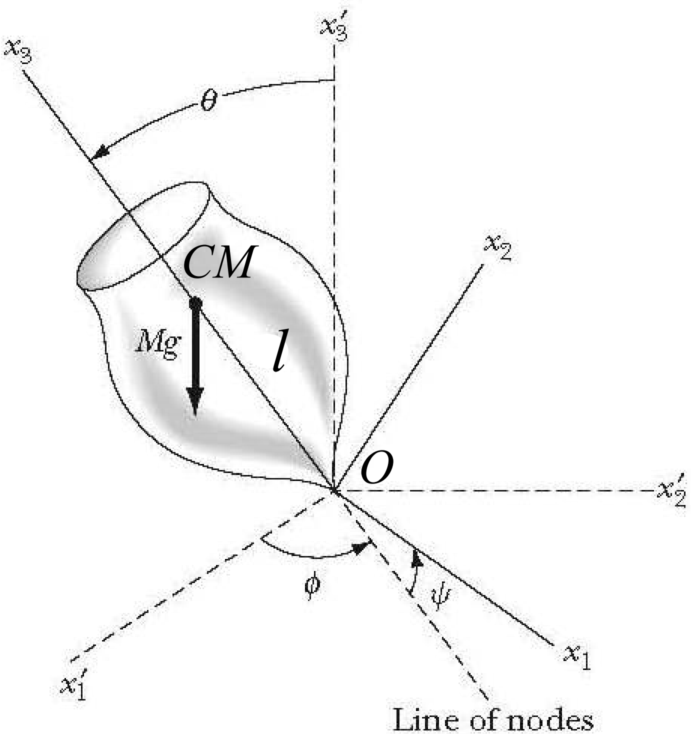
$$\oplus \quad \omega_2^2 = \dot\phi^2 \cos^2\psi \sin^2\theta + \dot\theta^2\sin^2\psi - \cancel{2\dot\phi\dot\theta\sin\psi\sin\theta\cos\psi}$$

$$\omega_1^2 + \omega_2^2 = \dot\phi^2\sin^2\theta \qquad + \dot\theta^2$$

Symmetric Top in an Uniform Gravity Field

So, we have $T = \frac{I_1}{2}\left(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2\right) + \frac{I_3}{2}\left(\dot{\phi}\cos\theta + \dot{\psi}\right)^2$

$$\Rightarrow \quad L = T - U = \frac{I_1}{2}\left(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2\right) + \frac{I_3}{2}\left(\dot{\phi}\cos\theta + \dot{\psi}\right)^2 - Mgl\cos\theta$$



Immediately, notice that both ϕ, ψ are cyclic !

This means that we immediately have the following two constants of motion:

$$\frac{\partial L}{\partial \dot{\psi}} = I_3\left(\dot{\phi}\cos\theta + \dot{\psi}\right) = p_\psi = const$$

$$\frac{\partial L}{\partial \dot{\phi}} = I_1\dot{\phi}\sin^2\theta + I_3\left(\dot{\phi}\cos\theta + \dot{\psi}\right)\cos\theta = p_\phi = const$$

$$= \left(I_1\sin^2\theta + I_3\cos^2\theta\right)\dot{\phi} + I_3\cos\theta\dot{\psi} = p_\phi$$

Symmetric Top in an Uniform Gravity Field

L does not depend on time explicit, so that the Jacobi integral h is another constant of motion.

Also, since the description of the orientation of the rigid body using the Euler angles (as the generalized coordinates) does not dep on t explicitly and U does not depend on $\dot{q}$, we should have $h = E = const$. Let check...

$$h = \dot{q}\frac{\partial L}{\partial \dot{q}} - L \qquad L = \frac{I_1}{2}\left(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2\right) + \frac{I_3}{2}\left(\dot{\phi}\cos\theta + \dot{\psi}\right)^2 - Mgl\cos\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = I_1\dot{\theta}$$

$$\frac{\partial L}{\partial \dot{\psi}} = I_3\left(\dot{\phi}\cos\theta + \dot{\psi}\right) \qquad \frac{\partial L}{\partial \dot{\phi}} = I_1 \sin^2\theta\,\dot{\phi} + I_3\left(\dot{\phi}\cos\theta + \dot{\psi}\right)\cos\theta$$

Symmetric Top in an Uniform Gravity Field

Substituting the partial derivatives of L into h, we have

$$h = \dot{q}\frac{\partial L}{\partial \dot{q}} - L = I_1\dot{\theta}^2 + I_3\left(\dot{\phi}\cos\theta + \dot{\psi}\right)\dot{\psi} + \left[I_1\sin^2\theta\,\dot{\phi} + I_3\left(\dot{\phi}\cos\theta + \dot{\psi}\right)\cos\theta\right]\dot{\phi}$$

$$-\frac{I_1}{2}\left(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2\right) - \frac{I_3}{2}\left(\dot{\phi}\cos\theta + \dot{\psi}\right)^2 + Mgl\cos\theta$$

$$= I_1\dot{\theta}^2 + I_3\left(\dot{\phi}\cos\theta + \dot{\psi}\right)\dot{\psi} + I_1\sin^2\theta\,\dot{\phi}^2 + I_3\left(\dot{\phi}\cos\theta + \dot{\psi}\right)\cos\theta\,\dot{\phi}$$

$$-\frac{I_1}{2}\dot{\theta}^2 - \frac{I_1}{2}\dot{\phi}^2\sin^2\theta - \frac{I_3}{2}\left(\dot{\phi}\cos\theta + \dot{\psi}\right)^2 + Mgl\cos\theta$$

$$= \frac{1}{2}I_1\dot{\theta}^2 + \frac{1}{2}I_1\sin^2\theta\,\dot{\phi}^2 + \frac{1}{2}I_3\left(\dot{\phi}\cos\theta + \dot{\psi}\right)^2 + Mgl\cos\theta$$

$$= \frac{1}{2}I_1\left(\omega_1^2 + \omega_2^2\right) + \frac{1}{2}I_3\omega_3^2 + U = E$$

✓

Symmetric Top in an Uniform Gravity Field

Now, continue with our analysis of the motion of the symmetric top using the Lagrangian:

$$\boldsymbol{\omega} = \begin{pmatrix} \dot{\phi}\sin\psi\sin\theta + \dot{\theta}\cos\psi \\ \dot{\phi}\cos\psi\sin\theta - \dot{\theta}\sin\psi \\ \dot{\phi}\cos\theta + \dot{\psi} \end{pmatrix}$$

We have the quantity $\left(\dot{\phi}\cos\theta + \dot{\psi}\right)$ equals to ω_3 !

→ so that the conservation of p_ψ also implies ω_3 being a constant.

$$I_3\omega_3 = I_3\left(\dot{\phi}\cos\theta + \dot{\psi}\right) = p_\psi$$

For convenience later, we will rename this constant (RHS) as $I_1 a$,

$$I_3\omega_3 = I_1 a \qquad (1)$$

or

$$I_3\cos\theta\,\dot{\phi} + I_3\dot{\psi} = I_1 a \qquad (2)$$

Symmetric Top in an Uniform Gravity Field

Similarly, we will rename the other constant p_ϕ to $I_1 b$ so that we have

$$\left(I_1 \sin^2\theta + I_3 \cos^2\theta\right)\dot{\phi} + I_3 \cos\theta \dot{\psi} = p_\phi = I_1 b \qquad (3)$$

$$I_3 \cos\theta\, \dot{\phi} + I_3 \dot{\psi} = I_1 a$$

Now, solving for $\dot{\psi}$ from Eq. (2), we have,

$$\dot{\psi} = \frac{I_1 a - I_3 \cos\theta\, \dot{\phi}}{I_3} \qquad (4)$$

Substituting this into Eq. (3), we then have,

$$\left(I_1 \sin^2\theta + I_3 \cos^2\theta\right)\dot{\phi} + \cos\theta\left(I_1 a - I_3 \cos\theta\, \dot{\phi}\right) = I_1 b$$

The blue terms cancel and gives,

$$\not{I_1} \sin^2\theta\, \dot{\phi} + \not{I_1} a \cos\theta = \not{I_1} b \quad \Rightarrow \quad \dot{\phi} = \frac{b - a\cos\theta}{\sin^2\theta}$$

Symmetric Top in an Uniform Gravity Field

$$\dot{\psi} = \frac{I_1 a - I_3 \cos\theta\,\dot{\phi}}{I_3}$$

- Substituting this expression for $\dot{\phi}$ into Eq. (4), we have,

$$\dot{\psi} = \frac{I_1}{I_3} a - \cos\theta \left(\frac{b - a\cos\theta}{\sin^2\theta} \right)$$

$$\dot{\phi} = \frac{b - a\cos\theta}{\sin^2\theta}$$

- So, using the two constants of motion for the problems, we have two explicit 1st order ODEs for two of the Euler angles ϕ, ψ in terms of θ.

- The next step is to try to write down an ODE for θ using the energy conservation equation,

$$\frac{I_1}{2}\left(\omega_1^2 + \omega_2^2\right) \qquad \frac{I_3}{2}\omega_3^2$$

$$E = T + U = \frac{I_1}{2}\left(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2\right) + \frac{I_3}{2}\left(\dot{\phi}\cos\theta + \dot{\psi}\right)^2 + Mgl\cos\theta$$

- Since we know both $\dot{\phi}, \dot{\psi}$ in terms of θ, this in fact is an ODE in θ only.

Symmetric Top in an Uniform Gravity Field

- But, actually, there is a short-cut…

- Recall that $\dot{\phi}\cos\theta + \dot{\psi} = \omega_3$ is a constant of the motion so that

$$E' \equiv E - \frac{I_3}{2}\omega_3^{\ 2} = \frac{I_1}{2}\left(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2\right) + Mgl\cos\theta = const$$

- Now, we substitute in $\dot{\phi}^2$

$$\dot{\phi}^2 = \frac{\left(b - a\cos\theta\right)^2}{\sin^4\theta}$$

$$\Rightarrow \quad E' = \frac{I_1}{2}\dot{\theta}^2 + \left(\frac{I_1}{2}\frac{\left(b - a\cos\theta\right)^2}{\sin^{\cancel{4}\,2}\theta}\cancel{\sin^2\theta} + Mgl\cos\theta\right)$$

Symmetric Top in an Uniform Gravity Field

- Rewriting this, we have the desired ODE for θ,

$$\frac{I_1}{2}\dot{\theta}^2 = E' - V_{eff}(\theta)$$

$$V_{eff}(\theta) = \frac{I_1}{2}\frac{(b - a\cos\theta)^2}{\sin^2\theta} + Mgl\cos\theta$$

- The direct method is to integrate this to get $\theta(t)$. Then, substitute it back into the ODEs for $\dot{\phi}, \dot{\psi}$ and integrate to get $\phi(t), \psi(t)$.

Symmetric Top in an Uniform Gravity Field

- A smarter alternative is to treat this as an effective 1D problem as we have done for the central force problems in Chapter 3.

$$E' = \frac{I_1}{2}\dot{\theta}^2 + V_{eff}(\theta) \qquad V_{eff}(\theta) = \frac{I_1}{2}\frac{(b - a\cos\theta)^2}{\sin^2\theta} + Mgl\cos\theta$$

- Before we proceed further, it will be helpful to briefly describe the physical meaning of the three Euler angles:

- 3rd Euler angle: $\dot{\psi}$ = spin about the body's symmetry axis
- 1st Euler angle: $\dot{\phi}$ = precession of the body's symmetry axis about the space x_3' ($\hat{\mathbf{z}}$)axis

(have seen in torque free case)

- 2nd Euler angle: $\dot{\theta}$ = nutation (bobbing up & down) of the body symmetry axis (this is new)

- So, our effective 1D treatment will tell us about this new behavior (nutation) !

Precession and Nutation for a Symmetric Top

- Similar to the effective 1D central force problem in Chapter 3, we can get a qualitative understanding of the system behavior by considering the geometric shape of $V_{eff}(\theta)$ and its relation to *E*'.

- Here is a plot of $V_{eff}(\theta)$:

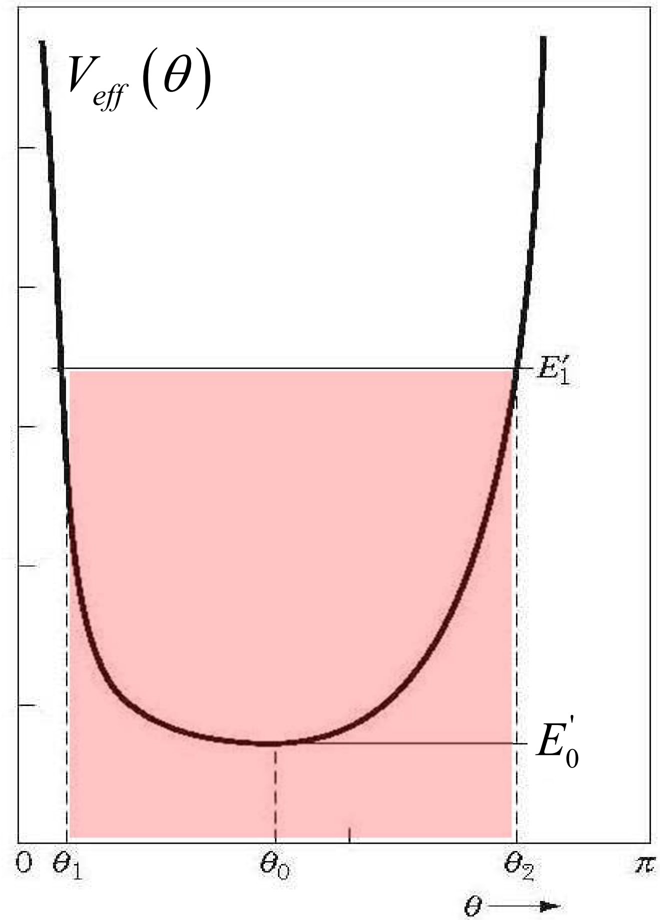
(actual shape will depend on p_ϕ, p_ψ)

- Since we need $\dot{\theta}^2 \geq 0$, for a given value of *E*', the physically allowed motion must have,

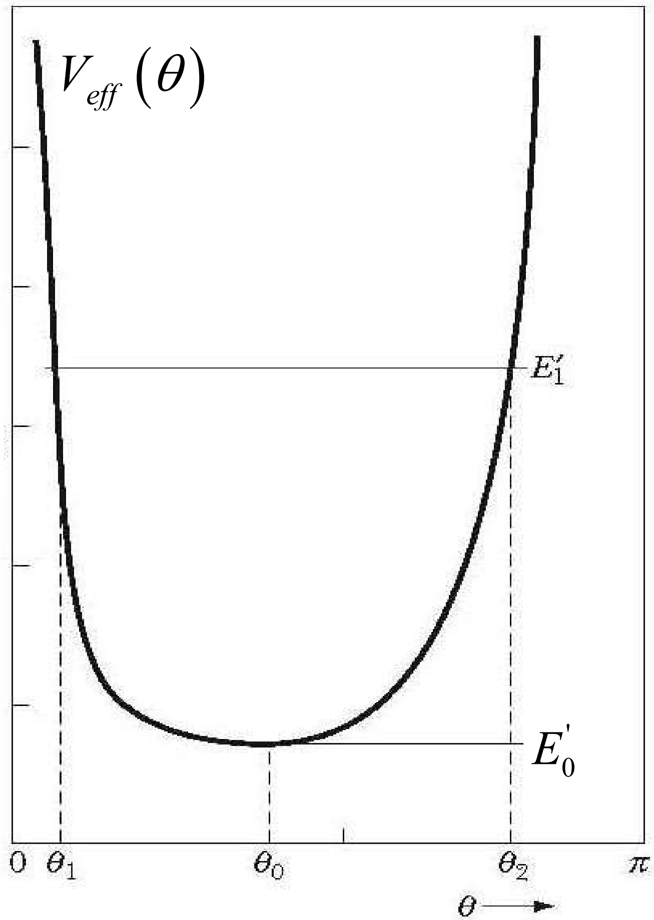
$$\frac{I_1}{2}\dot{\theta}^2 = E' - V_{eff}(\theta) \geq 0$$

- So, motion must remains within $\theta_1 \leq \theta \leq \theta_2$)

(physically allowed region for $E' = E_1'$)



Precession and Nutation for a Symmetric Top



Overview:

1. There is a minimum value of

$$E' = E_0' = V_{eff}(\theta_0)$$

for which there is only ONE allowed value for $\theta = \theta_0$ (pure precession).

2. For larger values of $E' > E_0'$ such as

$$E' = E_1'$$

θ is bounded between 2 values:

$$\theta_1 \le \theta \le \theta_2$$

This is the case of nutation. We will look at these two situations closer next.

Precession for a Symmetric Top

Case 1: $\theta = \theta_0$ with $E' = E_0' = V_{eff}(\theta_0)$ @ minimum (no nutations)

- The body axis x_3 is tilted at a fixed value $\theta = \theta_0$ wrt the fixed axis (no nutation)
- θ_0 can be determined by setting,

$$\left.\frac{dV_{eff}(\theta)}{d\theta}\right|_{\theta=\theta_o} = 0$$

$$V_{eff}(\theta) = \frac{I_1}{2}\left(\frac{b - a\cos\theta}{\sin\theta}\right)^2 + Mgl\cos\theta$$

- Evaluating the derivative of $V_{eff}(\theta)$ and setting it to zero, we have,

$$I_1\left(\frac{b - a\cos\theta_0}{\sin\theta_0}\right)\left(\frac{\sin\theta_0(-a)(-\sin\theta_0) - (b - a\cos\theta_0)\cos\theta_0}{\sin^2\theta_0}\right) - Mgl\sin\theta_0 = 0$$

$$I_1(b - a\cos\theta_0)\left(\frac{a\sin^2\theta_0 - (b - a\cos\theta_0)\cos\theta_0}{\sin^3\theta_0}\right) - Mgl\sin\theta_0 = 0$$

Precession for a Symmetric Top

Case 1: Continuing...

$$I_1\left(b-a\cos\theta_0\right)\left(\frac{a\sin^2\theta_0-\cos\theta_0\left(b-a\cos\theta_0\right)}{\sin^3\theta_0}\right)-Mgl\sin\theta_0=0$$

$$I_1\left[a\sin^2\theta_0\left(b-a\cos\theta_0\right)-\cos\theta_0\left(b-a\cos\theta_0\right)^2\right]-Mgl\sin^4\theta_0=0$$

- Let $\gamma=I_1\left(b-a\cos\theta_0\right)$, we can then write,

$$\left(\frac{\cos\theta_0}{I_1}\right)\gamma^2-\left(a\sin^2\theta_0\right)\gamma+Mgl\sin^4\theta_0=0$$

- Solving for γ, we then have,

$$\gamma=\frac{aI_1\sin^2\theta_0\pm\sqrt{\left(aI_1\sin^2\theta_0\right)^2-4\cos\theta_0\left(I_1Mgl\sin^4\theta_0\right)}}{2\cos\theta_0}$$

Precession for a Symmetric Top

Case 1:

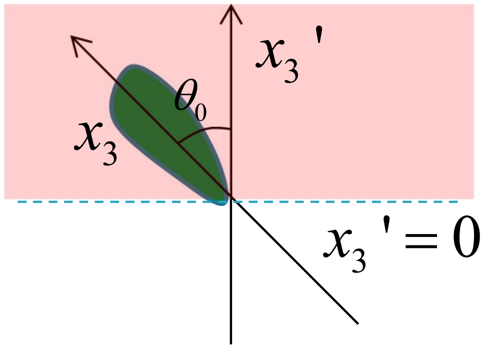
$$\gamma = \frac{aI_1 \sin^2\theta_0 \pm \sqrt{\left(aI_1 \sin^2\theta_0\right)^2 - 4\cos\theta_0 \left(I_1 Mgl \sin^4\theta_0\right)}}{2\cos\theta_0}$$

- Factoring out the common factor $\frac{aI_1 \sin^2\theta_0}{2\cos\theta_0}$, we have,

$$\gamma = \frac{aI_1 \sin^2\theta_0}{2\cos\theta_0}\left[1 \pm \sqrt{1 - \frac{4Mgl\cos\theta_0}{I_1 a^2}}\right]$$

- Recall that $\gamma = I_1\left(b - a\cos\theta_0\right)$ and it must be a real number for physically realizable situations and this allows us to make certain conclusions on the solution θ_0

Precession for a Symmetric Top



Two subcases:

Case 1a: $\theta_0 < \pi/2$ The tip of the top is above the horizontal plane ($x_3' = 0$ in the fixed frame).

→ This means that $\cos\theta_0 > 0$ and for γ to be real, we need to have

$$1 - \frac{4Mgl\cos\theta_0}{I_1 a^2} \geq 0 \qquad \left(\gamma = \frac{aI_1 \sin^2\theta_0}{2\cos\theta_0}\left[1 \pm \sqrt{1 - \frac{4Mgl\cos\theta_0}{I_1 a^2}}\right] \right)$$

Recall that we have $I_1 a = I_3\omega_3$. Substituting and rewriting, we have,

$$1 \geq \frac{4I_1 Mgl\cos\theta_0}{(I_3\omega_3)^2} \Rightarrow \omega_3 \geq \frac{2}{I_3}\sqrt{I_1 Mgl\cos\theta_0} = \omega^*$$

⇒ This means that in order to maintain a steady precession at a *fixed* tilt $\theta_0 < \pi/2$, ω_3 must be fast enough, i.e., $\omega_3 \gg \omega^*$ (fast top)

Precession for a Symmetric Top

$$\gamma_{\mp} = \frac{aI_1 \sin^2\theta_0}{2\cos\theta_0}\left[1 \mp \sqrt{1 - \frac{4Mgl\cos\theta_0}{I_1 a^2}}\right]$$

Case 1a: $\theta_0 < \pi/2$

- Since the equation for γ is quadratic, there will be two solution for the precession rate:

$$\gamma = I_1\left(b - a\cos\theta_0\right)$$

$$\dot{\phi}_{0\mp} = \frac{b - a\cos\theta_0}{\sin^2\theta_0} = \frac{\gamma_{\mp}}{I_1 \sin^2\theta_0}$$

To simplify γ, we will assume ω_3 to be large (fast top), then

$$\sqrt{1 - \frac{4I_1 Mgl\cos\theta_0}{\left(I_3\omega_3\right)^2}} \simeq 1 - \frac{2I_1 Mgl\cos\theta_0}{\left(I_3\omega_3\right)^2}$$

$$\Rightarrow \quad \gamma_{\mp} = \frac{I_3\omega_3 \sin^2\theta_0}{2\cos\theta_0}\left[1 \mp \left(1 - \frac{2I_1 Mgl\cos\theta_0}{\left(I_3\omega_3\right)^2}\right)\right]$$

Precession for a Symmetric Top

$$\gamma_{\mp} = \frac{I_3\omega_3 \sin^2\theta_0}{2\cos\theta_0}\left[1 \mp \left(1 - \frac{2I_1 Mgl\cos\theta_0}{(I_3\omega_3)^2}\right)\right]$$

Case 1a: $\theta_0 < \pi/2$

$$\gamma_{\mp} = \begin{cases} - \rightarrow \dfrac{I_3\omega_3 \sin^2\theta_0}{2\cos\theta_0}\dfrac{2I_1 Mgl\cos\theta_0}{(I_3\omega_3)^2} = \dfrac{I_1\sin^2\theta_0 Mgl}{I_3\omega_3} \\ + \rightarrow \dfrac{I_3\omega_3\sin^2\theta_0}{2\cos\theta_0}\left(2 - \dfrac{2I_1 Mgl\cos\theta_0}{(I_3\omega_3)^2}\right) = \dfrac{I_3\omega_3\sin^2\theta_0}{\cos\theta_0} \end{cases}$$

(small compare to 2)

fast spinning top $\omega_3 \gg \omega^*$

$$\Rightarrow \quad \dot{\phi}_{0\mp} = \frac{\gamma_{\mp}}{I_1\sin^2\theta_0} = \begin{cases} \dfrac{Mgl}{I_3\omega_3} \\ \dfrac{I_3\omega_3}{I_1\cos\theta_0} \end{cases}$$

Slow precession

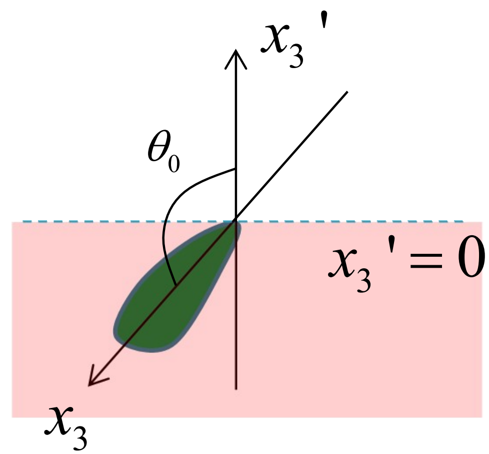
Fast precession

(note: the fast precession is independent of gravity g)

fast spinning top $\omega_3 \gg \omega^*$

Precession for a Symmetric Top

Case 1b: $\theta_0 > \pi/2$ The tip of the top is below the horizontal plane ($x_3' = 0$ in the fixed frame). Top is supported by a point support (show).



→ Here $\cos\theta_0 < 0$ and γ will always be real !

$$\gamma = \frac{aI_1 \sin^2\theta_0}{2\cos\theta_0}\left[1 \pm \sqrt{1 - \frac{4Mgl\cos\theta_0}{I_1 a^2}}\right] \in \mathrm{Re}$$

$$1 + \frac{4Mgl\left|\cos\theta_0\right|}{I_1 a^2} > 0$$

⇒ No special condition on ω_3 .

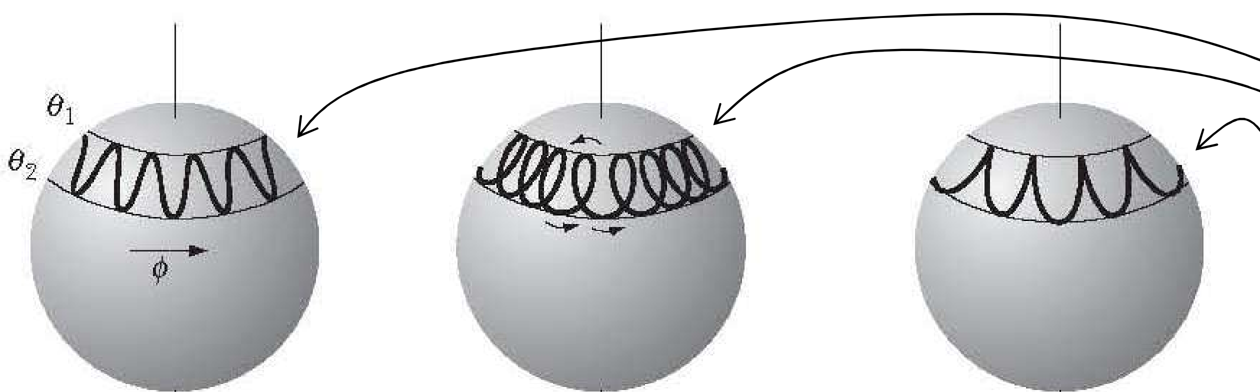
With top started with initial condition $\theta_0 > \pi/2$, it will remain below the horizontal plane and precesses around the fixed axis x_3' .

Nutation for a Symmetric Top

Case 2: $\theta_1 \le \theta \le \theta_2$ General situation with $E' > E_0' = V_{eff}(\theta_0)$. The body axis x_3 will blob up and down as it precesses around the fixed axis x_3' (nutation).

The precession rate of the body axis x_3 is described by: $\dot{\phi} = \dfrac{b - a\cos\theta}{\sin^2\theta}$

So, depending on *a* and *b*, $\dot{\phi}$ might or might not change sign… and we have the following three cases: (*a* and *b* are proportional to the 2 consts of motion: p_ψ, p_ϕ)



$b > a$

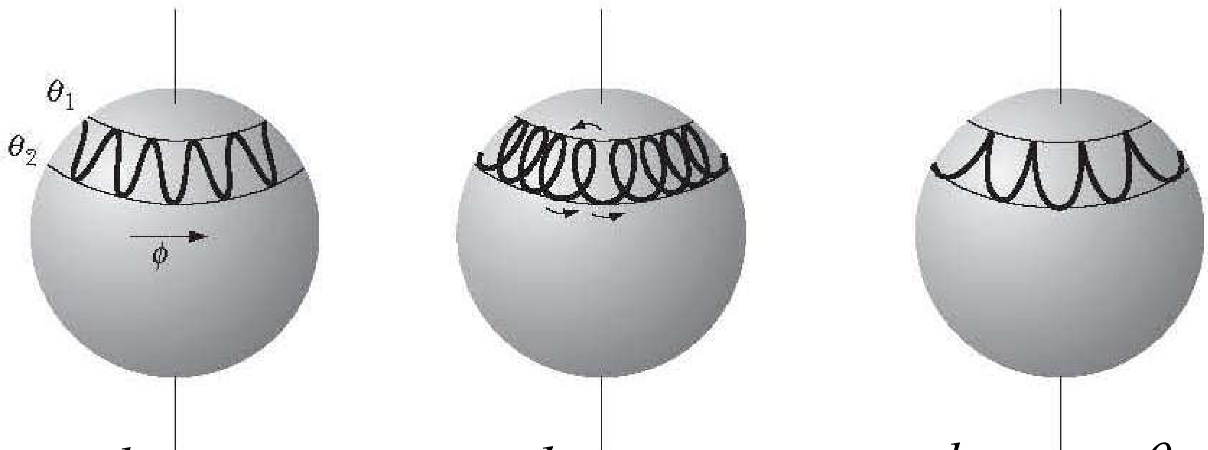
$b < a$
(reversal possible)

$b = a\cos\theta_1$
$\dot{\phi}(\theta_1) = 0$

Paths trace by the body axis x_3 as it precesses around the fixed axis x_3' (vertical).

Nutation for a Symmetric Top

The precession rate of the body axis x_3 is described by: $\dot{\phi} = \dfrac{b - a\cos\theta}{\sin^2\theta}$



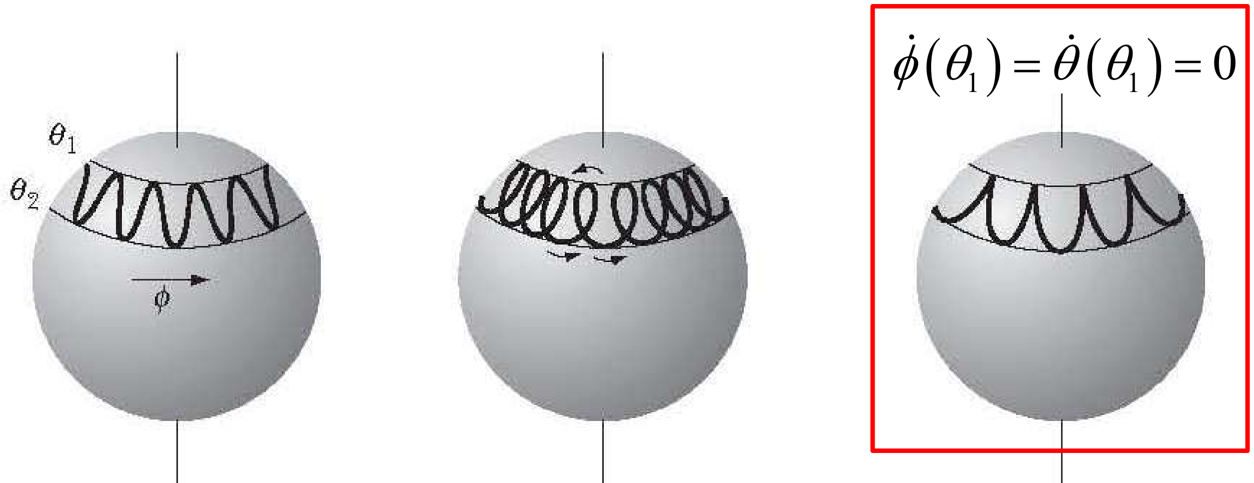
$b > a$

$b < a$ (reversal possible)

$b = a\cos\theta_1$
$\dot{\phi}(\theta_1) = 0$

https://www.youtube.com/watch?v=5Sn2J1Vn4zU

Nutation for a Symmetric Top



The last situation actually corresponds to the typical situation when the symmetric top is initially spinning about its body axis x_3 fixed at a given direction $\theta = \theta_0$ wrt to the fixed $x_3{}'$ axis. Then, the top is released.

- To be explicit, the initial conditions for this situation are:

$$\theta(0) = \theta_0,\ \dot{\phi}(0) = \dot{\theta}(0) = 0, \text{ and } \dot{\psi} \neq 0$$

- We will continue to assume the fast top condition $\omega_3 \gg \omega^*$ so that θ can remain above the $x_3{}' = 0$ plane ($\theta < \pi/2$).

Nutation for a Symmetric Top

$$\dot{\phi}(\theta_1) = \dot{\theta}(\theta_1) = 0$$

- At $t = 0$, these ICs implies that

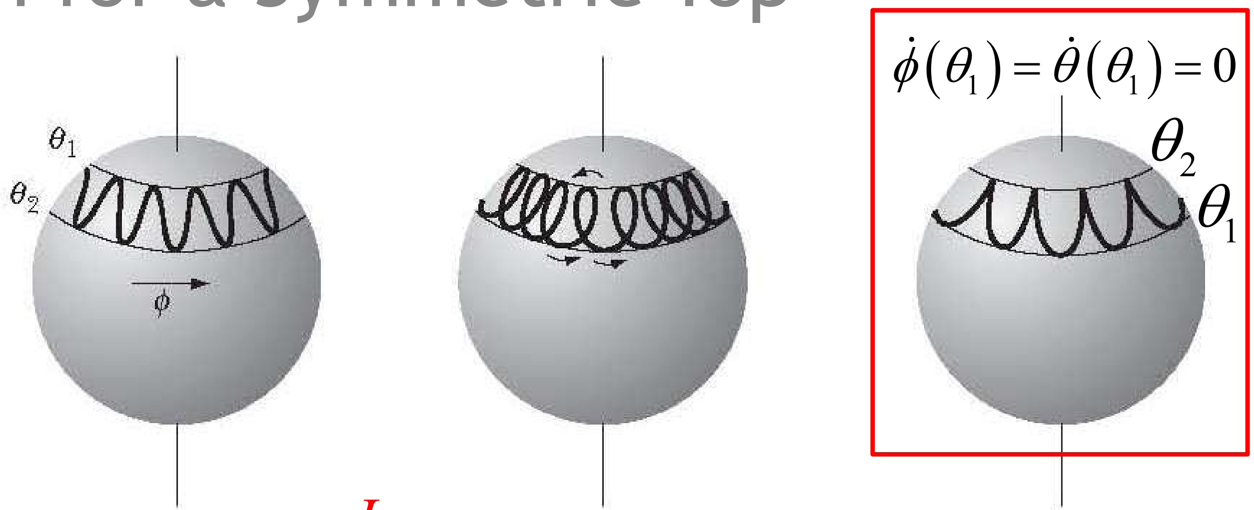
$$E' = E - \frac{I_3}{2}\omega_3{}^2 = \frac{I_1}{2}\left(\cancel{\dot{\phi}^2 \sin^2\theta_0} + \cancel{\dot{\theta}^2}\right) + Mgl\cos\theta_0$$

$$E' = Mgl\cos\theta_0$$

- At any subsequent time $t > 0$, energy remains conserved and we need to have,

$$E' = Mgl\cos\theta_0 = \frac{I_1}{2}\left(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2\right) + Mgl\cos\theta$$

Nutation for a Symmetric Top



$$Mgl\cos\theta_0 = \frac{I_1}{2}\left(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2\right) + Mgl\cos\theta$$

- Note that the KE terms (quadratic) on the RHS can't never be negative

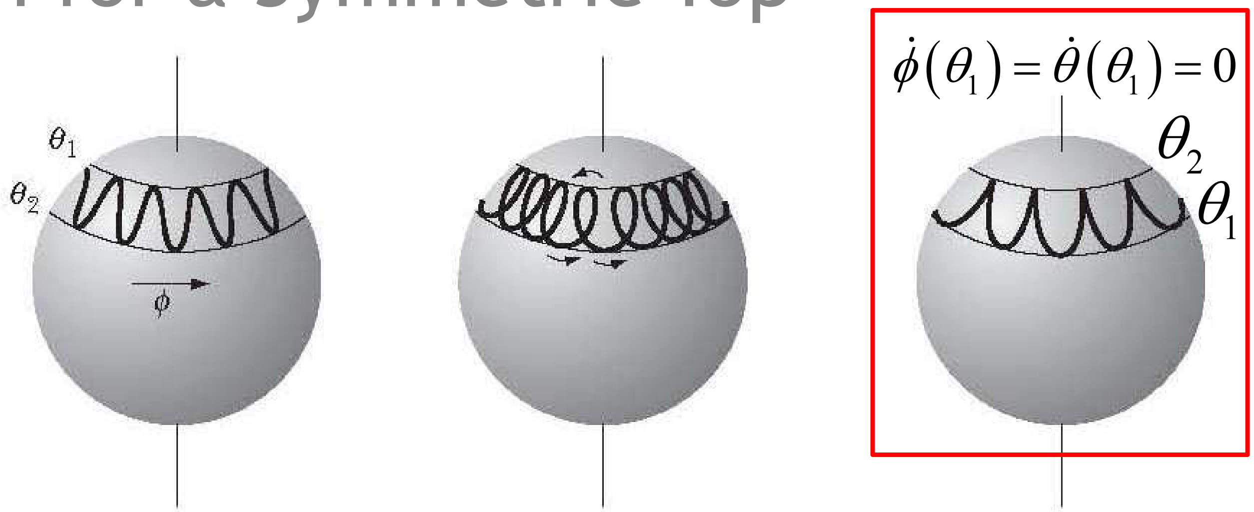
- So, as soon as $\dot{\phi}, \dot{\theta}$ begin to differ from their initial *zero* value,

→ The potential energy term must decrease correspondingly

➡ $\cos\theta < \cos\theta_0 \left(recall\, 0 \le \theta \le \pi/2\right)$ ➡ $\theta > \theta_0$ (top nutates down)

→ The initial value θ_0 will also be the smallest θ value (θ_2) that it can have (cusp of the curve).

Nutation for a Symmetric Top



→ When released in this manner, the top always starts to fall and continues to fall until it reaches the other turning point (θ_1). Then, it will rise again and repeat the cycle as it precesses around the fixed axis $x_3\,'$.

- We will now go further by estimating the range and frequency of this nutation (for a fast top) in the following analysis.

Nutation for a Symmetric Top

In order to do that we will start with the energy equation again,

$$E' = \frac{I_1}{2}\dot{\theta}^2 + \frac{I_1}{2}\frac{\left(b - a\cos\theta\right)^2}{\sin^2\theta} + Mgl\cos\theta$$

Recall: $\dot{\phi} = \dfrac{b - a\cos\theta}{\sin^2\theta}$

Rescaling our energies: $\alpha = 2E'/I_1$ and $\beta = 2Mgl/I_1$, we have,

$$\alpha = \dot{\theta}^2 + \frac{\left(b - a\cos\theta\right)^2}{\sin^2\theta} + \beta\cos\theta$$

Then, defining a new variable $u = \cos\theta$, we can transform the equation into ...

$$\dot{u}^2 = \left(1 - u^2\right)\left(\alpha - \beta u\right) - \left(b - au\right)^2$$

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quick check…

Nutation for a Symmetric Top

With $u = \cos\theta$, we have the following relations,

$$\dot{u} = \sin\theta\,\dot{\theta} \quad \rightarrow \quad \dot{\theta} = \frac{\dot{u}}{\sin\theta} \qquad\qquad \sin^2\theta = 1 - u^2$$

Substituting them into our previous equation: $\alpha = \dot{\theta}^2 + \dfrac{(b - a\cos\theta)^2}{\sin^2\theta} + \beta\cos\theta$

$$\alpha = \frac{\dot{u}^2}{1-u^2} - \frac{(b-au)^2}{1-u^2} + \beta u$$

$$\alpha - \beta u = \frac{\dot{u}^2 - (b-au)^2}{1-u^2}$$

$$\dot{u}^2 = (1-u^2)(\alpha - \beta u) - (b - au)^2$$

Note: the four constants of motion:

$$\alpha \rightarrow E' \qquad a \rightarrow p_\psi$$

$$\beta \rightarrow Mg \qquad b \rightarrow p_\phi$$

Nutation for a Symmetric Top

Now, with our initial conditions with the top spinning at $\theta = \theta_0 < \dfrac{\pi}{2}$ (as a fast top, i.e.,), $\omega_3 \gg \omega^*$ we would like to estimate the range of nutation.

With our initial conditions: $\theta(0) = \theta_0, \dot{\phi}(0) = \dot{\theta}(0) = 0$, and $\dot{\psi} \neq 0$

$$\dot{\phi}(0) = \frac{b - a\cos\theta_0}{\sin^2\theta_0} = 0 \rightarrow \boxed{b = au_0}$$

$$\dot{\phi}(0) = \dot{\theta}(0) = 0 \rightarrow E' = Mgl\cos\theta_0 \rightarrow \boxed{\alpha = \beta u_0}$$

$$E' = \frac{I_1}{2}\dot{\theta}^2 + \frac{I_1}{2}\dot{\phi}^2 + Mgl\cos\theta$$

$$\alpha = 2E'/I_1$$

$$\beta = 2Mgl/I_1$$

Nutation for a Symmetric Top

With the energy conservation equation:

$$\dot{u}^2 = \left(1-u^2\right)\left(\alpha-\beta u\right)-\left(b-au\right)^2$$

Substitute in $b = au_0$ and $\alpha = \beta u_0$ (ICs) into the above equation, we have,

$$\dot{u}^2 = \left(1-u^2\right)\left(\beta u_0-\beta u\right)-\left(au_0-au\right)^2$$

$$= \left(1-u^2\right)\beta\left(u_0-u\right)-a^2\left(u_0-u\right)^2$$

$$u = \cos\theta$$

$$\dot{u}^2 = \left(u_0-u\right)\left[\beta\left(1-u^2\right)-a^2\left(u_0-u\right)\right] = f\left(u\right)$$

Governing eq for nutations

For turning points (equilibrium points), the RHS $f\left(u\right)$ must equal to zero. We already know $u = u_0$ is a solution (our IC). The other solution u_1 must come from

$$\left(1-u^2\right)-\frac{a^2}{\beta}\left(u_0-u\right)=0 \quad \text{specifically,} \quad \left(1-u_1^2\right)-\frac{a^2}{\beta}\left(u_0-u_1\right)=0$$

Nutation for a Symmetric Top

Now, adding and subtracting $u_0^{\ 2}$ and $2u_0u_1$, we can rewrite the previous eq,

$$u_0^{\ 2} - 2u_0u_1 - \left(1 - u_1^{\ 2}\right) + \frac{a^2}{\beta}\left(u_0 - u_1\right) - 2u_0^{\ 2} + 2u_0u_1 + u_0^{\ 2} = 0$$

$$u_0^{\ 2} - 2u_0u_1 + u_1^{\ 2} - 1 + \frac{a^2}{\beta}\left(u_0 - u_1\right) - 2u_0^{\ 2} + 2u_0u_1 + u_0^{\ 2} = 0$$

Nutation for a Symmetric Top

Now, adding and subtracting $u_0{}^2$ and $2u_0u_1$, we can rewrite the previous eq,

$$u_0{}^2 - 2u_0u_1 - \left(1 - u_1{}^2\right) + \frac{a^2}{\beta}\left(u_0 - u_1\right) - 2u_0{}^2 + 2u_0u_1 + u_0{}^2 = 0$$

$$u_0{}^2 - 2u_0u_1 + u_1{}^2 - 1 + \frac{a^2}{\beta}\left(u_0 - u_1\right) - 2u_0{}^2 + 2u_0u_1 + u_0{}^2 = 0$$

$$\left(u_0 - u_1\right)^2 + \frac{a^2}{\beta}\left(u_0 - u_1\right) - 2u_0\left(u_0 - u_1\right) - \left(1 - u_0{}^2\right) = 0$$

$$\left(u_0 - u_1\right)^2 + \left(\frac{a^2}{\beta} - 2u_0\right)\left(u_0 - u_1\right) - \left(1 - u_0{}^2\right) = 0$$

Nutation for a Symmetric Top

Defining $x = u_0 - u$ and $\hat{x} = u_0 - u_1$ as the (max) range of nutation,

$$(u_0 - u_1)^2 + \left(\frac{a^2}{\beta} - 2u_0\right)(u_0 - u_1) - \left(1 - u_0^{\ 2}\right) = 0$$ can be written as,

$$\hat{x}^2 + \left(\frac{a^2}{\beta} - 2u_0\right)\hat{x} - \left(1 - u_0^{\ 2}\right) = 0$$

$$u_0 = \cos\theta_0$$

$$\hat{x}^2 + \left(\frac{a^2}{\beta} - 2\cos\theta_0\right)\hat{x} - \left(1 - \cos^2\theta_0\right) = 0$$

By solving the above quadratic equation for $\hat{x}$, we can get an estimate for the max range of the nutation.

Nutation for a "Fast" Top

Now, applying the "fast" top condition, $\omega_3 \gg \frac{2}{I_3}\sqrt{I_1 Mgl\cos\theta_0} = \omega^*$

or $\quad \frac{1}{2}\left(I_3\omega_3\right)^2 \gg I_1 Mgl\cos\theta_0 \qquad \left(KE_{rot} \gg PE_{gravity}\right)$

For an initially small tilt ($\cos\theta_0 \simeq 1$ or $\theta_0 \simeq 0^\circ$), this condition implies that

$$\frac{a^2}{\beta} = \left(\frac{I_3}{I_1}\right)\frac{I_3\omega_3^{\ 2}}{2Mgl} \gg 1 \quad \Rightarrow \quad \frac{a^2}{\beta} - 2\cos\theta_0 \cong \frac{a^2}{\beta}$$

$a = I_3\omega_3/I_1$

$\beta = 2Mgl/I_1$

Nutation for a "Fast" Top

Now back to the equation for the max range of nutation $\hat{x}$,

$$\hat{x}^2 + \left(\frac{a^2}{\beta} - 2\cos\theta_0 \right) \hat{x} - \sin^2\theta_0 = 0$$

Assuming $\hat{x}$ to be small and dropping the $\hat{x}^2$ term and with $\frac{a^2}{\beta} - 2\cos\theta_0 \cong \frac{a^2}{\beta}$

$$\hat{x}^2 + \left(\frac{a^2}{\beta} - 2\cos\theta_0 \right) \hat{x} - \sin^2\theta_0 \quad \simeq \frac{a^2}{\beta} \hat{x} - \sin^2\theta_0 = 0$$

$$\Rightarrow \quad \boxed{\hat{x} \simeq \frac{\beta}{a^2} \sin^2\theta_0 = \frac{2 I_1 M g l}{(I_3 \omega_3)^2} \sin^2\theta_0 \qquad (*)}$$

Nutation for a “Fast” Top

$$\hat{x} \simeq \frac{2I_1 Mgl}{\left(I_3\omega_3\right)^2}\sin^2\theta_0$$

From this solution, we can conclude that the maximum range of nutation will in general decrease as

$$\hat{x} \sim \frac{1}{\omega_3{}^2}$$

Thus, the faster the top spun, the less is the nutation !

Frequency of Nutation for a "Fast" Top

For a "fast" top, we can also estimate the frequency of its nutation…

As, we have seen for a fast top, when ω_3 is large and the range of nutation $\widehat{x}$ is small so that $u = \cos\theta \simeq \cos\theta_0 = u_0$:

$$\Rightarrow \quad 1 - u^2 \simeq 1 - u_0^{\ 2} = 1 - \cos^2\theta_0 = \sin^2\theta_0$$

Also, using our expression for $\widehat{x}$ for a "fast" top, Eq. (*), $\left(\widehat{x} = \dfrac{\beta}{a^2}\sin^2\theta_0 \right)$

$$\Rightarrow \quad \beta\left(1 - u^2\right) \simeq \beta\sin^2\theta_0 \simeq a^2\widehat{x}$$

Frequency of Nutation for a "Fast" Top

Substituting $\beta\left(1-u^2\right)\simeq a^2\widehat{x}$ into our expression for $f(u)$, and converting u into x, i.e., $x=u_0-u$, we have,

$$f(u)=\dot{u}^2=\left(u_0-u\right)\left[\beta\left(1-u^2\right)-a^2\left(u_0-u\right)\right]$$

$$\Downarrow$$

$$f(u)=\dot{x}^2=x\left[a^2\widehat{x}-a^2x\right]=a^2x\left(\widehat{x}-x\right)=a^2\left(-x^2+x\widehat{x}\right)$$

Letting $y=x-\dfrac{\widehat{x}}{2}$ (shifting origin to the mid-range of the nutation), we have,

$$\dot{y}^2=a^2\left(-x^2+x\widehat{x}-\frac{\widehat{x}^2}{4}+\frac{\widehat{x}^2}{4}\right)=a^2\left(-\left(x-\frac{\widehat{x}}{2}\right)^2+\frac{\widehat{x}^2}{4}\right)$$

$$\Rightarrow\qquad \dot{y}^2=a^2\left(-y^2+\frac{\widehat{x}^2}{4}\right)$$

Frequency of Nutation for a "Fast" Top

Upon differentiation , this differential equation reads,

$$\frac{d}{dt}\left[\dot{y}^2 = a^2\left(-y^2 + \frac{\hat{x}^2}{4}\right)\right] \Rightarrow \quad 2\dot{y}\ddot{y} = -a^2\left(2y\dot{y}\right)$$

Simplifying, we immediately get our desired result

$$\boxed{\ddot{y} = -a^2 y}$$

This describes a harmonic motion for the nutation as measured with respect to the mid-point of its range and the frequency of oscillation is,

$$\boxed{a = \frac{I_3}{I_1}\omega_3\left(= p_\psi\right)}$$

So, the frequency of nutation is faster if the top is spun faster initially $\left(\omega_3 \uparrow\right)$.